

Hydrodynamic modes of conducting liquid in random magnetic field

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March 1, 2016

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Hydrodynamics of plasma in the random magnetic field is considered, which is characterized by the second moment of magnetic induction. Equations of ideal magnetic hydrodynamics in such field are received for an adiabatic process. It is shown that in Euler Equation a new variable is second moment of magnetic induction enters, for which is received time Equation on the basis of Maxwell Equations in the magnetohydrodynamic approximation. The one dimension plane waves in this system are studied and the values of their phase velocities are received.

KEYWORDS: magnetic hydrodynamics, random field, second moment of magnetic induction, one-dimension plane waves, transversal velocity of sound.

Introduction

In magnetohydrodynamics (MHD) an environment is considered as single liquid and is described by the local values of density ρ , pressure P and velocity \vec{v} . Except for these cleanly hydrodynamic values the state of environment is characterized usual by magnetic induction \vec{B} ([1] p. 8), which has some unzeroing value in the equilibrium state. We will suppose, that the magnetic field is a accidental value with the second moment of magnetic induction \vec{B} different from a zero [2], which is not taken to the square of the first. Maxwell tensor of tensions ([1] p. 16) quadratic on the magnetic field is included in standard Euler Equation, for which we will build time Equation in MHD approximation. The offered generalization of MHD will allow to apply a theory to the systems in the accidental magnetic field, such as space plasma [3] and plasma of solid.

1 Equations of ideal MHD in the accidental field

We will write out standard Equations of ideal MHD, ignoring all dissipative effects (viscosity, heat conductivity and electric resistance) ([1] p. 19-24). Thus MHD effects show up most brightly. Continuity equation is

$$\partial_t \rho + \text{div} \rho \vec{v} = 0, \quad (1)$$

Euler equation is

$$d\rho \vec{v}/dt = -\nabla P - [\vec{B}, \vec{j}]/c, \quad (2)$$

here it is taken into account, that an environment unmagnetic and strength of magnetic field coincides with induction $\vec{H} = \vec{B}$. In addition, we will write down Maxwell Equations in MHD case for magnetic induction as accidental value

$$\text{rot} \vec{B} = 4\pi \vec{j}/c, \quad (3)$$

$$\partial_t \vec{B} = \text{rot}[\vec{v}, \vec{B}]. \quad (4)$$

At the construction of Eq. 4 it is taken into account, that at slow motions of environment electrons have time to be displaced toward the increase electric potential so that gradient of this potential will appeal to the zero. Thus the electric field in the own frame of reference ([4] p.89) equal to the zero, i.e. $\vec{E} = -[\vec{v}, \vec{B}]/c$. We will put an electric current from 3 in 2 and we will receives the law of momentum saving in a form

$$\partial_t (\rho v_i) + \partial_k \pi_{ik} = 0, \quad (5)$$

here denotations for derivative $\partial/\partial x_k = \partial_k$ and for stream of momentum tensor $\pi_{ik} = \rho v_i v_k + P \delta_{ik} - (B_i B_k - B^2 \delta_{ik}/2)/4\pi$ are entered. We will be interested only in small oscillations in the given system. It allows producing linearization on small amplitude deviations from the equilibrium values. Then

$$\pi_{ik} = \delta_{ik} \left((\partial P/\partial \rho)_s \rho + (\partial P/\partial s)_\rho s \right) - (\langle B_i B_k \rangle - \langle B_l B_l \rangle \delta_{ik}/2)/4\pi, \quad (6)$$

where deviations of correlation moments of the field from their equilibrium values is considered also having the first order of smallness. Apparently, Maxwell tensor of tensions enters in 5 which in the MHD approximation is fully determined by the second moment $\langle B_i B_k \rangle$. Time equation for the indicated moment we will receives from 4, multiplying on B_k in that spatio-time point and making symmetrization. After averaging on accidental phases ([5] p. 439) and linearization we have equation

$$\partial_t \langle B_i B_k \rangle = \eta_{iklmnp} \langle B_l B_m \rangle_0 \partial_n v_p, \quad (7)$$

where tensor $\eta_{iklmnp} = \delta_{ip} \delta_{km} \delta_{ln} + \delta_{im} \delta_{kp} \delta_{ln} - \delta_{il} \delta_{km} \delta_{np} - \delta_{im} \delta_{kl} \delta_{np}$ is entered. We will neglect by thermal fluctuations. In tensor of momentum stream 6 enters deviation of entropy. For simplicity we will take interest in adiabatic processes, then

$$\partial_t v_i + \partial_k \{ \delta_{ik} v_s^2 \rho - (\langle B_i B_k \rangle - \langle B_l B_l \rangle \delta_{ik}/2)/4\pi \} / \rho_0 = 0. \quad (8)$$

Here ρ_0 is equilibrium value of mass density, $v_s^2 = (\partial P/\partial \rho)_s$ is adiabatic velocity of sound in a nonmagnetized liquid. Also make linearization of Eq. 1

$$\partial_t \rho + \rho_0 \partial_i v_i = 0. \quad (9)$$

System of equations 9, 8 and 7 for variables ρ , v_i and $\langle B_i B_k \rangle$ is closed.

2 Adiabatic one-dimension waves of small amplitude

We will consider one-dimension waves, we will directs coordinate axis x_3 along direction of distribution. Lets all MHD values depend only of x_3 and t . Let the constant magnetic field have isotropic centered second moment, and also, selected direction for first moment. According to the done suppositions about statistics of the magnetic field the equilibrium value of correlation moment is

$$\langle B_l B_m \rangle_0 = \langle B_0^2 \rangle \delta_{lm}/3 + B_{0l} B_{0m} = \text{const.}$$

Without limitation of generality we can choose an axis $\vec{x}_1 \perp \vec{B}_0$,

i.e. $\vec{B}_0 = (0, B_0 \sin \theta, B_0 \cos \theta)$, where θ is angle between the first moment of the constant field and the wave. Equation 7 is symmetric on tensor indexes i and k , that is why contains 6 equations for component of symmetric tensor $\langle B_i B_k \rangle$. It is comfortably to represent ten equations of the system 7 - 9 in a matrix form

$$\partial_t \Psi_\alpha + Z_{\alpha\beta} \partial_3 \Psi_\beta = 0. \quad (10)$$

Here the vector of state

$$\Psi = (\rho, v_1, v_2, v_3, \langle B_1 B_1 \rangle, \langle B_1 B_2 \rangle, \langle B_1 B_3 \rangle, \langle B_2 B_2 \rangle, \langle B_2 B_3 \rangle, \langle B_3 B_3 \rangle) \quad (11)$$

and matrix with next nonzero components

$$\begin{aligned} Z_{14} &= \rho_0, Z_{27} = Z_{39} = -2Z_{45} = -2Z_{48} = 2Z_{410} = -1/4\pi\rho_0, Z_{41} = v_s^2/\rho_0, \\ Z_{54} &= 2\langle B_0^2 \rangle/3, Z_{84} = 2\langle B_0^2 \rangle/3 + 2B_0^2 \sin^2 \theta, \\ Z_{62} &= -2Z_{83} = Z_{94} = 2\langle B_0^2 \rangle/3 + 2B_0^2 \sin \theta \cos \theta, \\ Z_{72} &= Z_{93} = -\langle B_0^2 \rangle/3 - B_0^2 \cos^2 \theta \end{aligned} \quad (12)$$

are entered. In a plane one dimension wave dependence of the state vector on a coordinate and time looks like ([1] p.49-55)

$$\Psi_\alpha = A_\alpha \exp(ikx_3 - i\omega t). \quad (13)$$

Substitution Eq. 13 in Eq. 10 gives

$$Z_{\alpha\beta} A_\beta = V A_\alpha, \quad (14)$$

where $V = \omega/k$ is phase velocity of wave, A_α is right eigenvector of matrix Z . It is comfortably to enter denotations for Alfvén velocity $v_A = B_0/\sqrt{4\pi\rho_0}$ [1], and also for similar on a form velocity arising up due to the centered moment different from a zero $v_t = \sqrt{\langle B_0^2 \rangle/12\pi\rho}$ [2]. Solving Eq. 14 by the standard way we find the eigenvalues V of matrix Z

$$\begin{aligned} V = \left\{ 0, 0, 0, 0, -\sqrt{v_t^2 + v_A^2 \cos^2 \theta}, \sqrt{v_t^2 + v_A^2 \cos^2 \theta}, \right. \\ \left. -\sqrt{(3v_t^2 + v_s^2 + v_A^2) - \sqrt{(v_A^2 - v_t^2 - v_s^2)^2 - 4v_A^2(v_t^2 + v_s^2) \cos^2 \theta}}/\sqrt{2}, \right. \\ \left. \sqrt{(3v_t^2 + v_s^2 + v_A^2) - \sqrt{(v_A^2 - v_t^2 - v_s^2)^2 - 4v_A^2(v_t^2 + v_s^2) \cos^2 \theta}}/\sqrt{2}, \right. \\ \left. -\sqrt{(3v_t^2 + v_s^2 + v_A^2) + \sqrt{(v_A^2 - v_t^2 - v_s^2)^2 - 4v_A^2(v_t^2 + v_s^2) \cos^2 \theta}}/\sqrt{2}, \right. \\ \left. \sqrt{(3v_t^2 + v_s^2 + v_A^2) + \sqrt{(v_A^2 - v_t^2 - v_s^2)^2 - 4v_A^2(v_t^2 + v_s^2) \cos^2 \theta}}/\sqrt{2} \right\}, \end{aligned} \quad (15)$$

and also eigenvectors, which have bulky expressions. Unspreading perturbations of one of diagonal elements of tensor of the field second moment and density, and also perturbation of component $\langle B_1 B_2 \rangle$ correspond to the values $V = 0$. Six spreading waves correspond to interacting the MHD and new sound waves. Fifth and sixth eigenvalues correspond to the mode of transversal velocity

oscillations along an axis x_1 (perpendicular to both the wave and field \vec{B}_0) and components of correlation moment $\langle B_1 B_2 \rangle$ and $\langle B_1 B_3 \rangle$. Oscillations of velocity along axes x_2 and x_3 , density of mass, components $\langle B_2 B_3 \rangle$ and diagonal components $\langle B_1 B_1 \rangle$ and $\langle B_2 B_2 \rangle$ correspond to the other eigenvalues. From the received solution in supposition of absence of correlations of the magnetic field $\langle B_0^2 \rangle = 0$ the standard MHD solution as Alfven and two magnetosound waves [1], [5] follows. If to assume that the constant magnetic field is isotropic $B_0 = 0$, we have the following solution:

$$V = \left\{ 0, 0, 0, 0, -v_t, v_t, -v_t, v_t, -\sqrt{v_s^2 + 2v_t^2}, \sqrt{v_s^2 + 2v_t^2} \right\}.$$

Fifth and sixth eigenvalues correspond to the mode of transversal oscillations of velocity along an axis x_1 and components $\langle B_1 B_3 \rangle$. The seventh and eighth eigenvalues correspond to the mode of transversal oscillations of velocity along an axis x_2 and components $\langle B_2 B_3 \rangle$. The last two eigenvalues correspond to the mode of longitudinal oscillations of velocity along an axis x_3 , densities of mass and diagonal components $\langle B_1 B_1 \rangle$ and $\langle B_2 B_2 \rangle$. That in relation to mass velocity there are two transversal modes of oscillations with velocity v_t and one longitudinal with velocity $v_l = \sqrt{v_s^2 + 2v_t^2}$. There is a situation in this sense fully similar to the sound modes in an isotropic solid ([2], [6] p. 124-128) and the centered second moment of magnetic field determines the "module of displacement" $\mu = \langle B_0^2 \rangle / 12\pi$.

3 Conclusions

- Evolution of magnetized conducting liquid (or plasma) with the accidental field in approximation of ideal hydrodynamics is studied. Magnetic field as accidental value is described by value of second moment of magnetic induction. The linear system of equations for the density of mass, velocity and tensor of the second moment magnetic induction is received which allowed to study adiabatic modes in this system.
- Three modes of hydrodynamic oscillations are found in supposition of presence of constant first and isotropic centered second moments of the magnetic field. These modes turn into the standard MHD modes in absence of constant centered second moment. In default of the constant first moment two transversal with coincident phase velocities and longitudinal sound modes are received.

This work was supported by the State Foundation for Fundamental Research of Ukraine (project No.25.2/102).

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